Particle Conjugation and the $1/N_C$ Corrections to g_A

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ABSTRACT

We impose the requirement that the isovector axial vector current for the soliton sector of the chiral quark model transforms correctly under particle conjugation. This forces us to choose an otherwise arbitrary ordering of collective space operators in such a way that the next–to–leading $1/N_C$ correction to g_A vanishes.

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Recently, the very interesting observation has been made [1]–[3] that the too small value of the neutron beta decay constant g_A predicted in many chiral soliton models might be dramatically improved by including subleading $1/N_C$ corrections. These calculations, however, require one to make a certain choice about the ordering of two collective space operators which appear in the expression for the current. An ambiguity arises when one replaces classical objects by quantum operators while doing the collective quantization. In the present note we show that particle conjugation invariance can provide useful operator ordering information. It implies that the next–to–leading $1/N_C$ correction to g_A should vanish in the context of collectively quantizing the static soliton configuration. We will use the chiral quark model [4] but the calculation is essentially identical to those in other models of this type [1]–[3]. More detailed aspects as well as a similar study for the isovector part of the magnetic moment of the nucleon will be presented elsewhere [5].

In chiral models it is useful to employ the non–linear parametrization $U = \exp\left(i\boldsymbol{\tau} \cdot \boldsymbol{\pi}/f_\pi\right)$ of the pion field, $\boldsymbol{\pi}$. The static soliton configuration is described in terms of the hedgehog *ansatz*

$$U_0(\boldsymbol{r}) = \exp\left[i\boldsymbol{\tau} \cdot \hat{\boldsymbol{r}}F(r)\right], \tag{1}$$

where the radial function $F(r)$ is obtained by minimizing the static energy functional while maintaining definite boundary conditions. The projection of the soliton onto physical states is accomplished by introducing [6] time dependent collective coordinates $A(t) \in SU(2)$ parametrizing the isospin (and/or spin) orientation of the hedgehog

$$U(\boldsymbol{r}, t) = A(t)U_0(\boldsymbol{r})A^\dagger(t). \tag{2}$$

Due to the hedgehog structure of U_0 a rotation in space is equivalent to a *right* transformation of $A(t)$, *i.e.* the spin operator represents the *right* generator in the collective space

$$[\boldsymbol{J}, A] = A\frac{\boldsymbol{\tau}}{2}. \tag{3}$$

The angular velocity $\boldsymbol{\Omega}$, defined by

$$\frac{i}{2}\boldsymbol{\tau} \cdot \boldsymbol{\Omega} = A^\dagger \frac{\partial}{\partial t}A, \tag{4}$$

measures the *right* transformation at the classical level. When quantizing $A(t)$ canonically, the spin is identified as the momentum conjugate to $\boldsymbol{\Omega}$

$$\boldsymbol{J} = \frac{\partial L}{\partial \boldsymbol{\Omega}} = \alpha^2 \boldsymbol{\Omega}, \tag{5}$$

where $L = L(A, \boldsymbol{\Omega})$ denotes the collective Lagrangian. The moment of inertia, α^2, is a functional of the chiral angle, $F(r)$ and is of order N_C.

In the chiral quark model [4] one considers a Dirac spinor field in the background field U

$$\mathcal{L}_q = \overline{\Psi} \left(i\slashed{\partial} - mU^{\gamma_5} \right) \Psi, \tag{6}$$

where m parametrizes the quark–pion coupling. The contribution of the quarks to the static energy functional is given by

$$E_q^{\text{cl}} = N_C \text{sgn}(B) \epsilon_{\text{val}}, \tag{7}$$

where ϵ_{val} denotes the lowest (in magnitude) eigenvalue of the Dirac Hamiltonian

$$h(F) = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m \left[\cos F(r) + i\gamma_5 \boldsymbol{\tau} \cdot \hat{\boldsymbol{r}} \, \sin F(r) \right]. \tag{8}$$

In (7) the sign of the baryon number carried by the soliton is included as an additional factor to accommodate the hole interpretation of the Dirac theory. Next the collective coordinates are taken to be time dependent. Transforming to the flavor rotating frame $\Psi \to A\Psi$ adds the Coriolis term $\boldsymbol{\tau} \cdot \boldsymbol{\Omega}/2$ as a perturbation to (8). Then ϵ_{val} is changed at second order yielding the quark part of the collective Lagrangian

$$L_q = -E_q^{\text{cl}} + \frac{1}{2}\text{sgn}(B)\alpha_q^2 \boldsymbol{\Omega}^2 \;, \qquad \alpha_q^2 = \frac{N_C}{2} \sum_{\mu \neq \text{val}} \frac{|\langle \mu | \tau_3 | \text{val} \rangle|^2}{\epsilon_{\text{val}} - \epsilon_\mu}, \tag{9}$$

while the wave–function acquires a so–called cranking correction to become [7]

$$\Psi_{\text{crank}} = A(t) \left\{ \Psi_{\text{val}} + \frac{1}{2} \sum_{\mu \neq \text{val}} \Psi_\mu \frac{\langle \mu | \boldsymbol{\tau} \cdot \boldsymbol{\Omega} | \text{val} \rangle}{\epsilon_{\text{val}} - \epsilon_\mu} \right\}. \tag{10}$$

Now, the contribution of the *quarks* to the isovector axial vector current A_i^a is obtained by substituting (10) into $\text{sgn}(B)N_C\overline{\Psi}\gamma_i\gamma_5(\tau^a/2)\Psi$. The factor $\text{sgn}(B)$ has to be carried along because this current is a Noether current of the functional (9). Furthermore $\boldsymbol{\Omega}$ is replaced

by $\boldsymbol{J}/\alpha^2$ according to (5) yielding the quark part of the axial current operator in the space of the collective coordinates

$$
\begin{aligned}
A_i^{(\mathrm{q})a} \;=\; & \mathrm{sgn}(B) N_C D_{ab} \Psi_{\mathrm{val}}^\dagger \sigma_i \frac{\tau^b}{2} \Psi_{\mathrm{val}} \\
& +\mathrm{sgn}(B)\frac{N_C}{2\alpha^2}\left[(1-\xi)J_j D_{ab}+\xi D_{ab}J_j\right]\sum_{\mu\neq\mathrm{val}}\frac{\langle\mathrm{val}|\tau_j|\mu\rangle}{\epsilon_{\mathrm{val}}-\epsilon_\mu}\Psi_\mu^\dagger\sigma_i\frac{\tau^b}{2}\Psi_{\mathrm{val}}. \\
& +\mathrm{sgn}(B)\frac{N_C}{2\alpha^2}\left[(1-\eta)D_{ab}J_j+\eta J_j D_{ab}\right]\sum_{\mu\neq\mathrm{val}}\frac{\langle\mu|\tau_j|\mathrm{val}\rangle}{\epsilon_{\mathrm{val}}-\epsilon_\mu}\Psi_{\mathrm{val}}^\dagger\sigma_i\frac{\tau^b}{2}\Psi_\mu.
\end{aligned}
\tag{11}
$$

Here ξ and η denote *a priori* undetermined parameters labeling possible orderings between $D_{ab}=\frac{1}{2}\,\mathrm{tr}\left(\tau_a A\tau_b A^\dagger\right)$ and $\boldsymbol{J}$. Their presence reflects the ambiguity occurring when going from the classical to the quantum description. As α^2 is of the order N_C the ambiguous terms are subleading.

We note that the eigenvalues ϵ_μ are degenerate with respect to the grand spin (*i.e.* total spin plus isospin) projection, M_μ. The associated sum introduces a kind of reduced matrix element T_μ

$$
\sum_{M_\mu}\langle\mathrm{val}|\sigma_3\tau_b|\mu,M_\mu\rangle\langle\mu,M_\mu|\tau_j|\mathrm{val}\rangle \;=\; iT_\mu\epsilon_{3bj}.
\tag{12}
$$

An explicit calculation shows that T_μ is real [5]. The current (11) then yields the quark contribution to g_A

$$
\begin{aligned}
g_A^{\mathrm{q}} = \langle 2A_3^{(\mathrm{q})3}\rangle_N \;=\; & \mathrm{sgn}(B)N_C\Bigg\{\Big\langle D_{3b}\Big\rangle_N\langle\mathrm{val}|\sigma_3\tau_b|\mathrm{val}\rangle \\
& +\frac{1-\xi-\eta}{2\alpha^2}\sum_{\mu\neq\mathrm{val}}\frac{iT_\mu\epsilon_{3bj}}{\epsilon_{\mathrm{val}}-\epsilon_\mu}\left[\Big\langle D_{3b}J_j\Big\rangle_N-\Big\langle J_j D_{3b}\Big\rangle_N\right]\Bigg\},
\end{aligned}
\tag{13}
$$

where the subscript, N, indicates a matrix element between nucleon states. Here it should be stressed that the expression in square brackets vanishes classically; however, in the quantum description $\boldsymbol{J}$ does not commute with the rotation matrices

$$
[J_i,D_{ab}]=i\epsilon_{ibc}D_{ac}.
\tag{14}
$$

Then one finds for (13)

$$
g_A^{\mathrm{q}} = \mathrm{sgn}(B)N_C\left\{\Big\langle D_{3b}\Big\rangle_N\langle\mathrm{val}|\sigma_3\tau_b|\mathrm{val}\rangle-\frac{1-\xi-\eta}{\alpha^2}\Big\langle D_{33}\Big\rangle_N\sum_{\mu\neq\mathrm{val}}\frac{T_\mu}{\epsilon_{\mathrm{val}}-\epsilon_\mu}\right\}.
\tag{15}
$$

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The $1/N_C$ correction depends on the parameters ξ and η and is hence ambiguous. In various quark soliton models it has been shown that the prescription $\xi = \eta = 0$, which is suggested by the form of the cranked wave–function (10), results in sizable corrections to g_A improving the agreement with the experimental data [1]–[2]. However, we will now argue that requiring the proper behavior of the axial current under particle conjugation symmetry requires us to choose an ordering prescription so that the matrix elements of the $1/N_C$ terms actually vanish.

The particle conjugation symmetry is most conveniently implemented as G–parity invariance [8]. This transformation represents charge conjugation followed by an isorotation of angle π around the y–axis. Under G–parity the pion field changes sign, $i.e.$

$$U(\boldsymbol{r}, t) \xrightarrow{G} U^\dagger(\boldsymbol{r}, t). \tag{16}$$

Noting (2), we see that G–parity reflection for the rotating soliton corresponds to

$$F(r) \xrightarrow{G} -F(r) \tag{17}$$

while keeping the collective coordinates, $A(t)$ unchanged. At the soliton level the G–parity reflection agrees with the particle conjugation because the baryon number density reverses its sign under the transformation (17), $i.e.$ $B \to -B$. This property is a consequence of the fact that the isoscalar vector current has negative G–parity. The effect of the transformation (17) on the Dirac Hamiltonian may be simplified by introducing the self–adjoint unitary transformation $\mathcal{J} = i\beta\gamma_5$ and noting that

$$h(-F) = -\mathcal{J}^\dagger h(F)\mathcal{J}. \tag{18}$$

From this one may read off the particle conjugation properties of the eigenvalues and eigenstates of the Dirac Hamiltonian (8):

$$\epsilon_\mu \xrightarrow{F \to -F} -\epsilon_\mu \qquad \text{and} \qquad |\mu\rangle \xrightarrow{F \to -F} \mathcal{J}|\mu\rangle, \tag{19}$$

where μ labels the particular eigenstate. Since $\mathcal{J}$ does not affect isospin it is obvious that the Lagrangian (9) is invariant under particle conjugation, $i.e.$ the mass of the anti–baryon is identical to that of the baryon. Furthermore, as $\mathcal{J}$ has negative parity, the parity of the

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anti–baryon is opposite to that of the baryon, as it should be. Since the isovector axial vector current A_μ^a, is directly related to the pion field via $PCAC$ we must have

$$A_\mu^a \xrightarrow{G} - A_\mu^a, \tag{20}$$

under G–parity reflection.

Now we have collected all necessary ingredients and may consider the effect of particle conjugation on g_A. The reduced matrix element T_μ in (12) is invariant under this transformation because $\mathcal{J}\sigma_i = \sigma_i \mathcal{J}$. Thus we find[a]

$$g_A^q \xrightarrow{F\to -F} \bar{g}_A^q = \mathrm{sgn}(\bar{B})N_C \left\{ \left\langle D_{3b} \right\rangle_{\bar{N}} \langle \mathrm{val}|\sigma_3 \tau_b|\mathrm{val}\rangle + \frac{1 - \xi - \eta}{\alpha^2} \left\langle D_{33} \right\rangle_{\bar{N}} \sum_{\mu \neq \mathrm{val}} \frac{T_\mu}{\epsilon_{\mathrm{val}} - \epsilon_\mu} \right\} \tag{21}$$

since the energies ϵ_μ reverse their signs. The leading term changes sign under particle conjugation since $\mathrm{sgn}(\bar{B}) = -\mathrm{sgn}(B)$. This agrees with the required transformation property of the axial current under G–parity (20). However, the $1/N_C$ correction does not change sign under (19). Stated otherwise, the result for g_A violates the G–parity reflection symmetry unless

$$1 - \xi - \eta \equiv 0. \tag{22}$$

Thus G–parity invariance implies an operator ordering which rules out the $1/N_C$ corrections to g_A. We stress that this conclusion has been obtained without specifying the explicit wave–functions of the baryon or the anti–baryon. The constraint (22) is merely a consequence of the fact that the leading and next–to–leading order in $1/N_C$ terms in the approximate expression (13) for g_A transform oppositely under particle conjugation. It is interesting to note that the hermitean ordering $\xi = \eta = 1/2$ satisfies the constraint (22).

Here we have been studying only the contributions of the *quarks* to g_A in the chiral quark model. The leading order term of the mesonic axial current contains only odd powers of $F(r)$ and no $1/N_C$ corrections appear. Thus the transformation property (20) is trivially satisfied for the total current A_μ^a.

Although we have only considered the chiral quark model as the simplest containing soliton solutions, our studies apply to more complicated ones as well. In the literature the

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[a]For taking matrix elements we may note $\langle p \uparrow |D_{3b}|p \uparrow\rangle = \langle p \uparrow |D_{33}|p \uparrow\rangle\delta_{3b} = \langle \bar{n} \uparrow |D_{33}|\bar{n} \uparrow\rangle\delta_{3b} = -1/3\delta_{3b}$.

chiral bag [2] and Nambu–Jona–Lasinio (NJL) [1] models have been considered in connection with the $1/N_C$ corrections to g_A. In both models matrix elements of the structure (11,12) are involved in the subleading terms for g_A. Since the moment of inertia carries mass dimension one, the function multiplying the dimensionless reduced matrix element must be odd in the energy eigenvalues ϵ_μ. Thus the corrections do not transform properly under particle conjugation and an operator ordering which makes these corrections vanish has to be adopted.

Similar ambiguities appear in the next–to–leading order in the $1/N_C$ expansion of the isovector part of the magnetic moment of the nucleon, μ_V. We have also considered the behavior of these corrections under particle conjugation [5]. For this observable too, the proper behavior requires an ordering so that the $1/N_C$ corrections are zero.

To summarize, we have seen that particle conjugation symmetry provides useful constraints on operator ordering ambiguities which may occur when quantizing the chiral soliton. We have seen explicitly that these constraints require an ordering so that the next–to–leading $1/N_C$ correction to the axial charge of the nucleon vanishes in the chiral quark model.

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